Neutrino oscillations

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1 Neutrino Oscillations: A Brief Introduction

The neutrinos are very uncanny particles. Besides unimaginably small cross-sections and masses they can change their identity. Imagine you have a source of neutrinos of only one flavor. Let's say at the initial point with coordinate $x_0 = (t = 0, \boldsymbol{x} = 0)$ we have an accelerator producing muon neutrinos (ν_μ) in a state $|\nu_\mu(x_0)\rangle$. We let them travel a distance L. At the distance L we put a particle detector. Normally one would expect to find only muon neutrinos coming from the beam by looking for muons appearing inside of it. But instead we find also other types of leptons (fig. 1). Electrons and sometimes even taons, if the beam energy is sufficient. This phenomenon is called the "oscillation". Unfortunately, there is no satisfying explanation of it in the field of classical physics. One has to know at least the basic principles of quantum mechanics.
For more detailed explanation it is necessary to know the standard model of weak and electromagnetic interactions, introduced in [1]. In it the lepton masses are generated by the coupling of lepton fields to Higgs field. After the spontaneous gauge symmetry breaking it appeares that the flavour states of quarks and leptons are different from the states with defined masses. This phenomenon is parametrized for quarks in the so-called Cabibbo-Kobayashi-Maskawa mixing matrix. Because the quarks are confined it is impossible to measure their oscillations. But with neutrinos the story is completely different. For the details see for example [2] or [3].

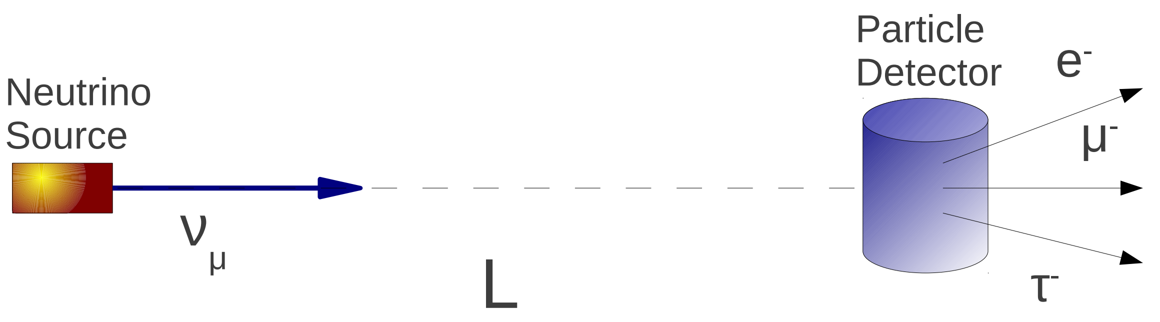

Figure 1: Muon neutrino oscillation experiment.

2 The Oscillation Mechanism

We would like to prepare the initial neutrino flavor state $|\nu_\alpha(x_0)\rangle$ in such a manner, that the probability of detecting a different state ν_β at coordinates $x1$ can be nonzero, e.g. $|\langle\nu_\beta(x_1)|\nu_\alpha(x_0)\rangle|^2 \neq 0$. The only way to do that is to assume all the neutrino *flavour* states are a mixture of different *mass* eigenstates, rather than pure states:

$$|\nu_\alpha\rangle = \sum_i U^*_{\alpha i}|\nu_i\rangle \tag{2.1}$$

Here it is convenient to explain the meaning of the "mass eigenstate". It is a state, which in the particle rest frame fulfills the following Schrödinger equation:

$$i\frac{\partial}{\partial\tau_i}|\nu_i(\tau_i)\rangle = m_i|\nu_i(\tau_i)\rangle \tag{2.2}$$

with τ_i being the particle's proper time and we are working in a $\hbar = c = 1$ unit system. Thus it is an eigenstate of the Hamiltonian a free particle with mass m_i. the solution to the above equation is simply:

$$|\nu_i(\tau_i)\rangle = e^{-im_i\tau_i}|\nu_i(\tau_i)\rangle \tag{2.3}$$

The reason, why we use the "proper time" will explain itself in a while. Now we need to discuss the mixing matrix properties. If one wants the state orthonormality condition to be fulfilled the mixing matrix $U_{i\alpha}$ can not be completely arbitrary. At the same point of space and time we would like to have:

$$\delta_{\alpha\beta} = \langle\nu_\beta(x)|\nu_\alpha(x)\rangle = \sum_{i,j} U^*_{\alpha i}U_{j\beta}\langle\nu_j|\nu_i\rangle = \sum_i U^*_{\alpha i}U_{i\beta} \tag{2.4}$$

thus the neutrino mixing matrix has to be unitary[1]. The most general form of the mixing matrix is as follows:

$$U = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \tag{2.5}$$

with the additional unitarity condition $\sum_{i=1}^3 |U_{\alpha i}|^2 = 1$. Let us calculate now the probability:

$$\begin{aligned} |\langle\nu_\beta(y)|\nu_\alpha(x)\rangle|^2 &= \left|\sum_{i,j} U^*_{\alpha i}U_{j\beta}\langle\nu_j(y)|\nu_i(x)\rangle\right|^2 = \left|\sum_i U^*_{\alpha i}U_{i\beta}\langle\nu_i(y)|\nu_i(x)\rangle\right|^2 = \\ &= \sum_{i,j} U^*_{\alpha i}U_{i\beta}U_{\alpha j}U^*_{j\beta}\langle\nu_i(y)|\nu_i(x)\rangle\langle\nu_j(y)|\nu_j(x)\rangle^* = \\ &= \sum_i |U_{\alpha i}|^2|U_{\beta i}|^2 + \sum_{i\neq j} U^*_{\alpha i}U_{i\beta}U_{\alpha j}U^*_{j\beta}\langle\nu_i(y)|\nu_i(x)\rangle\langle\nu_j(y)|\nu_j(x)\rangle^* . \end{aligned} \tag{2.6}$$

[1] This is, however, the truth only if there are no more lepton families, than predicted by the Standard Model.

What we need now is the neutrino mass eigenstate propagation amplitude. The solution 2.3 will be handy to write down the amplitude the particle rest frame:

$$\langle \nu_i(0)|\nu_i(\tau_i)\rangle \tag{2.7}$$

From the Lorentz invariance:

$$m_i\tau_i = p_\mu(y-x)^\nu = E_i(y^0 - x^0) - \boldsymbol{p_i}(\boldsymbol{y}-\boldsymbol{x}) \equiv Et - \boldsymbol{pL} \tag{2.8}$$

A convenient choice of the coordinate system is the one, in which the distance L is taken along the neutrino beam direction, e.g:

$$m_i\tau_i = E_i t - p_i L \tag{2.9}$$

thus the desired propagation amplitude will be:

$$\langle \nu_i(0)|\nu_i(\tau_i)\rangle = e^{-i(E_i t - p_i L)}. \tag{2.10}$$

The neutrinos produced by a source are coming in wave packets constructed from mass eigenstates, whose propagation is described by the above-mentioned formula. The oscillation phenomenon is described by the interference of propagation amplitudes of different mass eigenstates, as one cane see in 2.6. If we assume, that the neutrino source is constant in time, then we measure something proportional to the time average of the probability formula. Now let us take a look at the time-dependence part:

$$\left\langle e^{-i(E_i - E_j)t}\right\rangle_t = 0 \tag{2.11}$$

unless $E_i = E_j$. Following this argument the oscillation should come from mass eigenstates having the same energy (approach by Stodolsky). If all $E_i = E_j$, then the interference terms will become:

$$\langle \nu_i(y)|\nu_i(x)\rangle \langle \nu_j(y)|\nu_j(x)\rangle^* = \exp(i(p_i - p_j)L) \tag{2.12}$$

Here we have to make another assumption: the energy of neutrinos is of the order of 10^6 eV ([MeV]) or grater, thus the particles are ultra-relativistic with their rest mass being of the order of 1 eV. One can then expand the formula connecting relativistic momentum and energy in powers of $\frac{E}{M}^2$:

$$p_i = \sqrt{E^2 - m_i^2} = E - \frac{m_i^2}{2E} + \mathcal{O}\left(\frac{m_i}{E}\right)^4. \tag{2.13}$$

we have arrived at a point, where it is possible to define the interference terms leading to oscillation probability:

$$|\langle \nu_\beta(y)|\nu_\alpha(x)\rangle|^2 = \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2 + \sum_{i\neq j} U^*_{\alpha i}U_{i\beta}U_{\alpha j}U^*_{j\beta}\exp(i\frac{(m_j^2 - m_i^2)L}{2E}) \tag{2.14}$$

[2]Actually the more popular approach, as seen for example in Particle Data Group listings, assumes the mass eigenstates to have the same momenta. The result are the same, which can be easily checked by the reader. But it is harder to show, how to keep the mass states coherent over time.

We arrive at the conclusion, that neutrinos must be massive in order to oscillate. Moreover, at least one of the masses must be different from the others. This is a very important and fundamental result. To make the further discussion more clear some elementary algebra is needed. Let us denote:

$$\Delta m_{ij}^2 = m_i^2 - m_j^2 \tag{2.15}$$

we shall go back into the oscillation probability formula:

$$\begin{aligned}
|\langle\nu_\beta(y)|\nu_\alpha(x)\rangle|^2 &= \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2 + \sum_{i\neq j} U_{\alpha i}^* U_{i\beta} U_{\alpha j} U_{j\beta}^* \exp(-i\frac{\Delta m_{ij}^2 L}{2E}) = \\
&= \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2 + 2\sum_{i>j} \Re\left[U_{\alpha i}^* U_{i\beta} U_{\alpha j} U_{j\beta}^* \exp(-i\frac{\Delta m_{ij}^2 L}{2E})\right] = \\
&= \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2 + 2\sum_{i>j} \Re\left[U_{\alpha i}^* U_{i\beta} U_{\alpha j} U_{j\beta}^*\right] \Re\left[\exp(-i\frac{\Delta m_{ij}^2 L}{2E})\right] + \\
&- 2\sum_{i>j} \Im\left[U_{\alpha i}^* U_{i\beta} U_{\alpha j} U_{j\beta}^*\right] \Im\left[\exp(-i\frac{\Delta m_{ij}^2 L}{2E})\right]
\end{aligned} \tag{2.16}$$

Here we will make use of the following identities:

$$\Re e^{i\alpha} = \cos(\alpha);\ \ \Im e^{i\alpha} = \sin(\alpha) \tag{2.17}$$

to obtain following probability:

$$\begin{aligned}
|\langle\nu_\beta(y)|\nu_\alpha(x)\rangle|^2 &= \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2 + 2\sum_{i>j} \Re\left[U_{\alpha i}^* U_{i\beta} U_{\alpha j} U_{j\beta}^*\right] \cos\left(-\frac{\Delta m_{ij}^2 L}{2E}\right) + \\
&- 2\sum_{i>j} \Im\left[U_{\alpha i}^* U_{i\beta} U_{\alpha j} U_{j\beta}^*\right] \sin\left(-\frac{\Delta m_{ij}^2 L}{2E}\right) = \\
&= \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2 + 2\sum_{i>j} \Re\left[U_{\alpha i}^* U_{i\beta} U_{\alpha j} U_{j\beta}^*\right] \cos\left(\frac{\Delta m_{ij}^2 L}{2E}\right) + \\
&+ 2\sum_{i>j} \Im\left[U_{\alpha i}^* U_{i\beta} U_{\alpha j} U_{j\beta}^*\right] \sin\left(\frac{\Delta m_{ij}^2 L}{2E}\right)
\end{aligned} \tag{2.18}$$

Another improvement comes from the identity:

$$\cos(\alpha) = 1 - 2\sin^2\left(\frac{\alpha}{2}\right) \tag{2.19}$$

thus:

$$\begin{aligned}
|\langle\nu_\beta(y)|\nu_\alpha(x)\rangle|^2 &= \sum_i |U_{\alpha i}|^2 |U_{\beta i}|^2 + 2\sum_{i>j} \Re\left[U_{\alpha i}^* U_{i\beta} U_{\alpha j} U_{j\beta}^*\right] + \\
&- 4\sum_{i>j} \Re\left[U_{\alpha i}^* U_{i\beta} U_{\alpha j} U_{j\beta}^*\right] \sin^2\left(\frac{\Delta m_{ij}^2 L}{4E}\right) + \\
&+ 2\sum_{i>j} \Im\left[U_{\alpha i}^* U_{i\beta} U_{\alpha j} U_{j\beta}^*\right] \sin\left(\frac{\Delta m_{ij}^2 L}{2E}\right)
\end{aligned} \tag{2.20}$$

After a straight forward manipulation and use of the mixing matrix unitarity we obtain the final formula:

$$
\begin{aligned}
|\langle\nu_\beta(y)|\nu_\alpha(x)\rangle|^2 &= \delta_{\alpha\beta} - 4\sum_{i>j}\Re\left[U^*_{\alpha i}U_{i\beta}U_{\alpha j}U^*_{j\beta}\right]\sin^2\left(\frac{\Delta m^2_{ij}L}{4E}\right) + \\
&+ 2\sum_{i>j}\Im\left[U^*_{\alpha i}U_{i\beta}U_{\alpha j}U^*_{j\beta}\right]\sin\left(\frac{\Delta m^2_{ij}L}{2E}\right)
\end{aligned}
\tag{2.21}
$$

The reader may ask now a question: why should we bother to get from 2.14 to 2.21? The above formula has a big advantage over the previous one. And more than one. First of all it is easy to see, that if $x = y$ there are no oscillations, which was not that apparent by looking on 2.14.
Second of all: all quantum theories are assumed do be CPT- invariant. Thus one should get the same physics after switching the particles with antiparticles (C), making a mirror image of the space (P) and reversing the time flow (T). Let us take a look at the anti-neutrino oscillation probability:

$$
P\left(\overline{\nu}_\alpha \to \overline{\nu}_\beta\right) \overset{CPT}{=} P\left(\nu_\beta \to \nu_\alpha\right) = P\left(\nu_\alpha \to \nu_\beta, U \to U^*\right) \tag{2.22}
$$

Thus:

$$
\begin{aligned}
P\left(\overline{\nu}_\alpha \to \overline{\nu}_\beta\right) &= \delta_{\alpha\beta} - 4\sum_{i>j}\Re\left[U^*_{\alpha i}U_{i\beta}U_{\alpha j}U^*_{j\beta}\right]^2\sin\left(\frac{\Delta m^2_{ij}L}{4E}\right) + \\
&\overset{!!!}{-} 2\sum_{i>j}\Im\left[U^*_{\alpha i}U_{i\beta}U_{\alpha j}U^*_{j\beta}\right]\sin\left(\frac{\Delta m^2_{ij}L}{2E}\right)
\end{aligned}
\tag{2.23}
$$

This means, that if the mixing matrix U is complex, the oscillation probabilities for neutrinos and antineutrinos can[3] be different. This has further, deeper, consequences. It means that in the neutrino physics CP symmetry can be possibly broken. This is a subject of very sophisticated experimental studies. Now we can divide the oscillation probability into two parts:

$$
\begin{aligned}
P\left(\nu_\alpha/\overline{\nu}_\alpha \to \nu_\beta/\overline{\nu}_\beta\right) &= \underbrace{\delta_{\alpha\beta} - 4\sum_{i>j}\Re\left[U^*_{\alpha i}U_{i\beta}U_{\alpha j}U^*_{j\beta}\right]\sin^2\left(\frac{\Delta m^2_{ij}L}{4E}\right)}_{CP-conserving} + \\
&+/- \underbrace{2\sum_{i>j}\Im\left[U^*_{\alpha i}U_{i\beta}U_{\alpha j}U^*_{j\beta}\right]\sin\left(\frac{\Delta m^2_{ij}L}{2E}\right)}_{CP-violating}
\end{aligned}
\tag{2.24}
$$

From the experimentalists point of view one can detect the neutrino oscillations in two ways:

1. Search for the appearance of $\nu_\alpha \neq \nu_\beta$ from a ν_α source. Done by searching for the charged leptons from CC neutrino interactions. (*appearance* experiment)

2. Search for lacks of known ν_α flux.(*disappearance* experiment)

[3] But do not have to, think for example about an overall phase of the type $e^{i\phi}$!

Both methods require very intense neutrino sources, big detectors and lots of time. We have to remember we are dealing with a very weak interaction type. An experimentalist has to also know where to put the detector in order to get the best result. After putting $\hbar$ and c in the oscillation formula:

$$\frac{\Delta m_{ij}^2 L}{4E} = 1.27 \Delta m_{ij}^2 (eV^2) L(km)/E(GeV) \tag{2.25}$$

The experiments sensitivity to measured $\Delta m_{ij}^2(eV^2)$ is governed by the $\frac{E(GeV)}{L(km)}$ fraction. Because the $\sin\left(\frac{\Delta m_{ij}^2 L}{4E}\right)$ has to reach reasonably large values, the approximate experiment sensitivity is given by:

$$\Delta m_{ij}^2(eV^2) \propto \frac{E(GeV)}{L(km)}. \tag{2.26}$$

It is also worthy to mention, that the neutrino oscillation experiments are able only to give the difference between squared masses, not the neutrino masses themselves. To see, how the neutrino mixing is described and measured one has to first find a convenient parametrization of U. Let us start from a simplified 2-neutrino example.

2.1 The Two-Neutrino Example

Let us assume that we have only two flavor and two neutrino mass eigenstates named ν_α, ν_β and ν_1, ν_2 respectively. In this case the most general neutrino mixing matrix may look as follows:

$$U = e^{i\phi} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} e^{i\delta} & 0 \\ 0 & 1 \end{pmatrix} \tag{2.27}$$

It is a most general 2x2 unitary matrix. Here $e^{i\phi}$ is an overall phase factor, which does not change the physics, so we may drop it. The next term is a general 2-dimensional rotation matrix and the last term is the so-called Majorana CP-violating phase. Notice, that for the rotation matrix is a convenient way of describing, how much of each mass eigenstate one finds in a flavor state and vice versa. Rotation angles are also used to describe the most general three neutrino mixing matrix. For neutrino flavor states we have the following equation:

$$\begin{pmatrix} \nu_\alpha \\ \nu_\beta \end{pmatrix} = \begin{pmatrix} e^{i\delta}\cos(\theta) & \sin(\theta) \\ -e^{i\delta}\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} e^{i\delta}\cos(\theta)\nu_1 + \sin(\theta)\nu_2 \\ -e^{i\delta}\sin(\theta)\nu_1 + \cos(\theta)\nu_2 \end{pmatrix} \tag{2.28}$$

From this one can calculate the ν_α disappearance and survival probabilities:

$$P(\nu_\alpha \to \nu_\beta) = \sin^2(2\theta)\sin^2\left(\frac{\Delta m_{12}^2 L}{4E}\right) \tag{2.29}$$
$$P(\nu_\alpha \to \nu_\alpha) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta m_{12}^2 L}{4E}\right)$$

Notice, that the Majorana phase does not change the oscillation probability in vacuum. In this case the neutrino and antineutrino oscillations are the same. This is also true when there are all three lepton families involved. Because there is only one squared mass difference in this simplified example, we shall call it Δm^2 for simplicity. The oscillation amplitude is governed by $\sin(2\Theta)$, whereas the positions of oscillation probability extreama is governed by $\frac{\Delta m^2 L}{4E}$.

2.2 Matter Effect

Let us now assume we have neutrino mass eigenstaes with the same momenta p. As it has been stated before there are two approaches to the neutrino oscillation; one with equal mass eigenstate energies, and another one with equal momenta. Both leading to the same result, with slightly different approach to what we call coherent mass eigenstates. In the vacuum we may write down the Hamiltonian describing the neutrino mass eigenstates:

$$H\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{2.30}$$

We may use an ultra-relativistic approximation to the neutrino energy $E_i = \sqrt{p^2 + m_i^2} \approx p + \frac{m_i^2}{2p} \approx p + \frac{m_i^2}{2E}$:

$$H\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \approx \begin{pmatrix} p + \frac{m_1^2}{2E} & 0 \\ 0 & p + \frac{m_2^2}{2E} \end{pmatrix}\begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{2.31}$$

What is interesting to us is the oscillation probability. We need to separate the non-diagonal part of this Hamiltonian:

$$\begin{aligned} \begin{pmatrix} p + \frac{m_1^2}{2E} & 0 \\ 0 & p + \frac{m_2^2}{2E} \end{pmatrix} &= \left[\left(p + \frac{m_1^2 + m_2^2}{4E}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{m_1^2 - m_2^2}{4E}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \\ &= H_0 + H_{osc.} \end{aligned} \tag{2.32}$$

We can drop out every part of the Hamiltonian, which does not give rise to the phase difference between our mass eigenstates. In this matrix notation this will be true to all the terms proportional to $\mathbf{1}_{2x2}$. Now we would like to proceed to the neutrino flavour basis and calculate the transition matrix elements:

$$\left\langle \nu_\beta \left| H^{flavour}_{osc.\beta\alpha} \right| \nu_\alpha \right\rangle = \sum_{i,j} U^*_{\alpha i} U_{j\beta} \left\langle \nu_j \left| H^{flavour}_{osc.\beta\alpha} \right| \nu_i \right\rangle \tag{2.33}$$

Using the unitarity argument we obtain a relation:

$$\begin{aligned} H^{flavour}_{osc.} &= U H_{osc.} U^{-1} = -\frac{\Delta m^2}{4E} U \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} U^{-1} = \\ &= -\frac{\Delta m^2}{4E}\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = \\ &= \frac{\Delta m^2}{4E}\begin{pmatrix} -\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \end{aligned} \tag{2.34}$$

Where the Majorana phases have been dropped as unimportant to the overall oscillation pattern. By introducing the matter one also introduces interactions. The neutrino interactions can be divided into two parts: the charge-current interactions mediated through $W^\pm$ bosons and the neutral-current interactions mediated through Z^0 bosons. We assume our two flavor eigenstates to be electron and muon and take into account the types of interactions, which leave a neutrino in a final state. We want the neutrino beam to propagate through matter and see the difference with vacuum. We are not interested in the case,

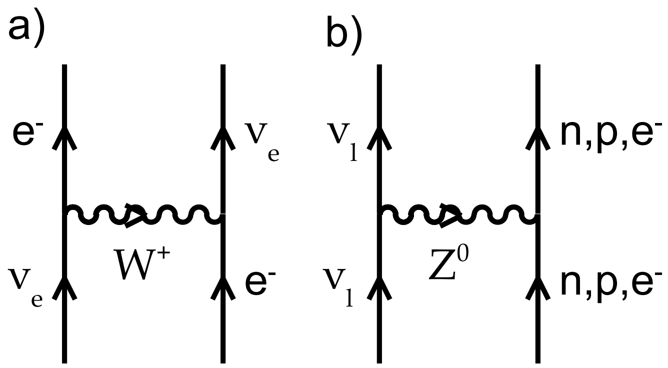

Figure 2: Neutrino interactions, which give rise to the matter potential.

when neutrinos disappear. Thus the only charged-current interaction to be taken here into account in normal matter is the one with electrons (see Fig.2(a)). Hence it does not affect the neutrinos in their muon state. The neutral -current interaction takes place on both electrons and atomic nuclei (see Fig.2(b)). We will introduce two potentials:

$$\begin{aligned} V_W &= +\sqrt{2} G_F N_e \ (- \text{ for } \bar{\nu}_e) \qquad (2.35) \\ V_Z &= -\frac{\sqrt{2}}{2} G_F N_p \ (+ \text{ for } \bar{\nu}) \end{aligned}$$

with G_F being the Fermi coupling constant and N_i the density of scattering centers in matter. They modify the flavor eigenstate oscillation Hamiltonian in the following way:

$$\begin{aligned} H_{osc.}^{flavour}(matter) &= \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} + V_W \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \qquad (2.36) \\ &= \frac{\Delta m^2}{4E} \begin{pmatrix} -\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} + \frac{V_W}{2} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right] = \\ &= \frac{\Delta m^2}{4E} \begin{pmatrix} -\left(\cos(2\theta) - \frac{2V_W E}{\Delta m^2}\right) & \sin(2\theta) \\ \sin(2\theta) & \left(\cos(2\theta) - \frac{2V_W E}{\Delta m^2}\right) \end{pmatrix} + \frac{V_W}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

The Z-boson potential is totally diagonal, thus it does not contribute to the oscillation pattern. We shall denote $x = \frac{2V_W E}{\Delta m^2} = \frac{2\sqrt{2} G_F N_e E}{\Delta m^2}$. Thus:

$$H_{osc.}^{flavour}(matter) = \frac{\Delta m^2}{4E} \begin{pmatrix} -(\cos(2\theta) - x) & \sin(2\theta) \\ \sin(2\theta) & (\cos(2\theta) - x) \end{pmatrix} \qquad (2.37)$$

The above equation can be a little bit more refined. First let us introduce new matter parameters:

$$\begin{aligned} \Delta m_M^2 &= \Delta m^2 \sqrt{\sin^2(2\theta) + (\cos(2\theta) - x)^2} \qquad (2.38) \\ \sin(2\theta_M) &= \frac{\sin(2\theta)}{\sqrt{\sin^2(2\theta) + (\cos(2\theta) - x)^2}} \end{aligned}$$

It is straightforward to show now, that:

$$\cos(2\theta_M) = \sqrt{1 - \sin^2(2\theta_M)} = \frac{(\cos(2\theta) - x)}{\sqrt{\sin^2(2\theta) + (\cos(2\theta) - x)^2}} \qquad (2.39)$$

and the matter Hamiltonian can be written down in a much more elegant form:

$$H^{flavour}_{osc.}(matter) \quad = \quad \frac{\Delta m^2_M}{4E}\begin{pmatrix} -\cos(2\theta_M) & \sin(2\theta_M) \\ \sin(2\theta_M) & \cos(2\theta_M) \end{pmatrix} \tag{2.40}$$

In this way we have obtained effective neutrino mixing angle and mass splitting in matter. When a neutrino travels through matter one has to substitute $(\Theta, \Delta m^2) \rightarrow (\Theta_M, \Delta m^2_M)$. Thus for the case of neutrinos traveling inside matter with approximately constant density:

$$\begin{aligned} P(\nu_e \rightarrow \nu_\mu)_M &= \sin^2(2\theta_M)\sin^2\left(\frac{\Delta m^2_M L}{4E}\right) \\ P(\nu_e \rightarrow \nu_e)_M &= 1 - \sin^2(2\theta_M)\sin^2\left(\frac{\Delta m^2_M L}{4E}\right) \end{aligned} \tag{2.41}$$

This result requires some more attention. First of all the size of the matter effect depends strictly on $x = \frac{2\sqrt{2}G_F N_e E}{\Delta m^2}$. The sign of this expression depends on the neutrino *mass hierarchy, e. g.* whether $m_1^2 > m_2^2$ or not. Second of all the potential changes sign if we change neutrinos to the antineutrinos. In the neutrino sector there exists a possibility for the CP violation. But because of the matter effect the experimentalists have to put a lot of effort to distinguish between the original CP violation and the matter neutrino-antineutrino effect! How big is this effect? The density of electrons in matter can be expressed as:

$$N_e = N_A Y_e \rho \tag{2.42}$$

with N_A being the Avogadro number, Y_e- the average number of electrons per nucleon and ρ -the matter density. If we assume, that $Y_e \approx \frac{1}{2}$ (the nuclei are symmetric in the proton and neutron numbers), then:

$$x \approx \frac{0.76 \times 10^{-5}\rho[g/cm^3]E[GeV]}{\Delta m^2[eV^2]} \tag{2.43}$$

Let us assume, that we want to work with the solar neutrinos. Their highest energy is around 12 $[MeV]$, thus around $12 \times 10^{-3}[GeV]$. The best value of Δm^2_{21} according to the Particle Data Group is $7.59 \times 10^{-5}[eV^2]$. thus for the maximum energy solar neutrinos one has:

$$x_{solar} \approx 1.2 * 10^{-2} \times \rho[g/cm^3] \tag{2.44}$$

Taking into account the values of $\sin^2(2\theta_{12}) = \sin^2(2\theta_{sol}) = 0.86$, $\sin(2\theta_{sol}) \approx 0.93$ and the formulas 2.39 the solar neutrino matter effect will be described by:

$$\begin{aligned} \Delta m^2_{sol.M} &= 7.59\sqrt{0.861 + (0.373 - 1.2 * 10^{-2}\rho[g/cm^3])^2} \times 10^{-5}[eV^2] \\ \sin(2\theta_{sol.M}) &= \frac{0.93}{\sqrt{0.861 + (0.373 - 1.2 * 10^{-2}\rho[g/cm^3])^2}} \end{aligned} \tag{2.45}$$

where the two neutrino approximation is fairly good, because $\sin^2(2\theta_{13})$ is known to be very small. Thus for the propagation in earth's crust, whose approximate density is 2.8 $[g/cm^3]$, the matter effect is as follows:

$$\begin{aligned} \Delta m^2_{sol.M} &\approx 7.5 \times 10^{-5}[eV^2] \\ \sin^2(2\theta_{sol.M}) &\approx 0.881 \end{aligned} \tag{2.46}$$

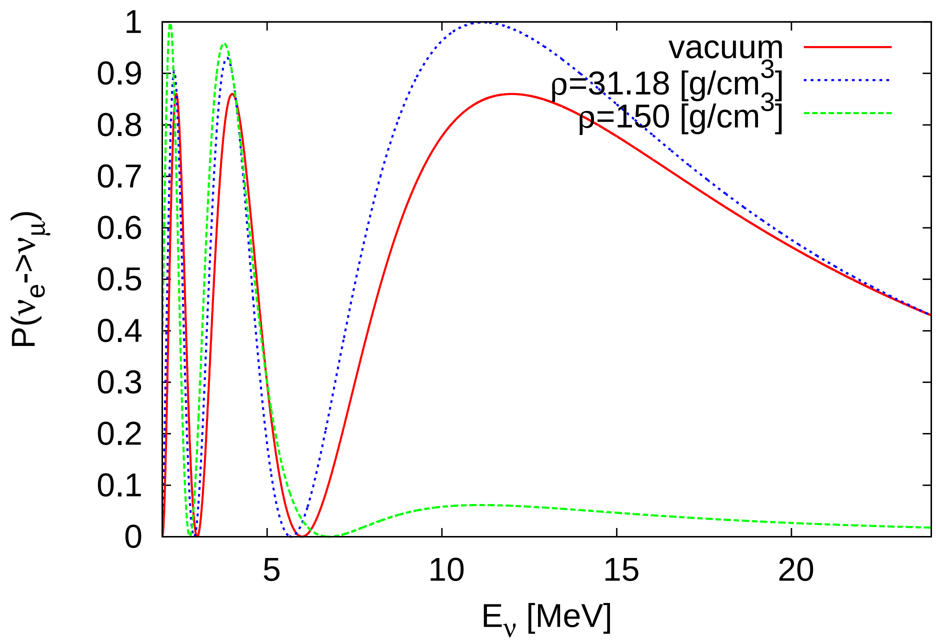

Figure 3: Matter effect example for $(\nu_e \to \nu_\mu)$ as a function of neutrino energy for the two neutrino oscillation case and fixed distance $L \approx 31[km]$.

The matter effect seems to be almost negligible in this case. The situation will change dramatically, if we take a look at their propagation in the Sun's core. The density inside of it is of the order of 150 $[g/cm^3]$. At this density level the matter effect becomes very large. This can be seen in the Fig. 3. The distance L has been picked to have a maximum oscillation probability at $E_\nu = 12[MeV]$. The red curve corresponds to the case of vacuum oscillations, which is a periodic function of $\frac{L}{E}$ and maximum oscillation probability equal to $\sin(2\Theta_{12})$. If we put the density level to the one in the Sun's core, the value of mixing angle will be $\sin^2(2\theta_M) \approx 0.3$ and the effective squared mass $\Delta m_M^2 \approx 13 \times 10^{-4}[eV^2]$. The large values of x can cause disappearance of the neutrino oscillations. This happens both due to the increase of matter density as well as the increase of neutrino energy. At some point the oscillation amplitude will start to disappear proportionally to E^{-2}. Another curious case appears when $x = \cos(2\theta)$. For the solar neutrinos this is an equivalent of density level around 31 $[g/cm^3]$. In this case one has $\sin(2\theta_{sol.M}) \approx 1$ and the oscillation/survival probabilities may reach 1 at given distances, as one can see in the blue curve. This is the so-called *resonance*. If we take into account the sign difference in x for neutrinos and antineutrinos, a clear example of false CP-violation will be obtained, as seen in Fig. 4. The density is the same, as in neutrino resonance case. As one can see, the difference between the neutrino and antineutrino oscillation patterns can be big. Thus the neutrino experimentalists have to be very careful while planning their experimental setups: not only they must find optimal distance/energy ration but also take multiple effects, like the matter influence, into account.

For further reading about the neutrino physics we recommend [2] and [4].

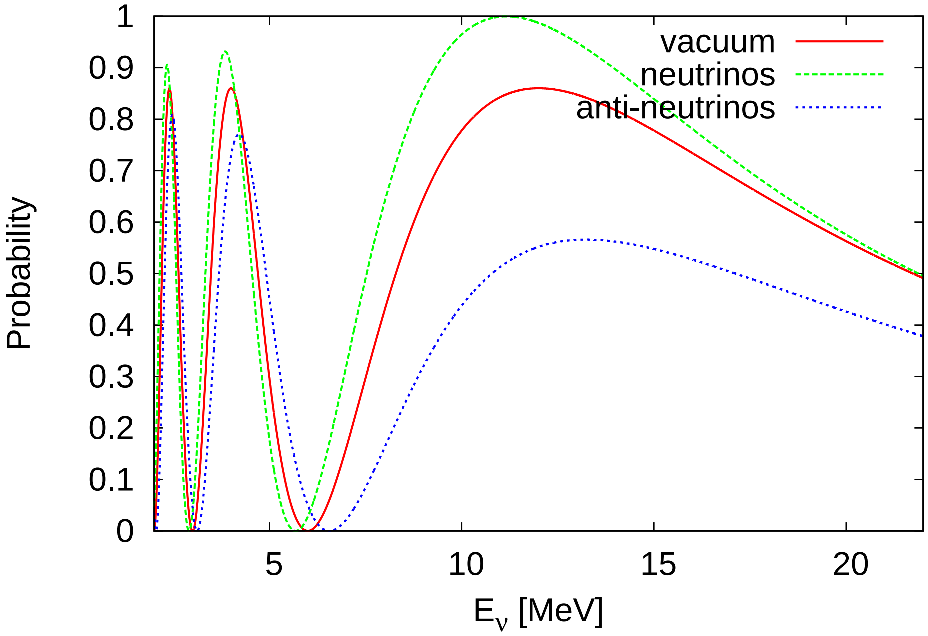

Figure 4: Matter effect example for $(\nu_e \to \nu_\mu)$ and $(\overline{\nu}_e \to \overline{\nu}_\mu)$ as a function of neutrino energy for the two neutrino oscillation case and fixed distance $L \approx 31[km]$.

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